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On the One-Dimensional Symmetric Two-Body Problem of Classical Electrodynamics

V. I. ZHDANOV

State Standards Committee, Moscow, USSR

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Abstract

The functional-differential equation which describes the one-dimensional symmetric motion of two charged particles in the framework of classical electrodynamics is considered. In the case of the charges of a like sign it is proved that the global solution exists and it is specified uniquely by the instantaneous initial data, if the classical energy at the initial moment is sufficiently small. In the case of the charges of opposite sign there are additional restrictions on the initial data. The estimates are given which allow one to obtain an approximate description of motion.

1. *Introduction*

We consider the functional-differential system (we put the speed of light $c = 1$) taking into account the retardation of interactions

$$\frac{\ddot{X}(t)}{(1-\dot{X}^2(t))^{3/2}} = \frac{k}{\tau^2(t)} \frac{1-\dot{X}(t-\tau(t))}{1+\dot{X}(t-\tau(t))} \tag{1.1}$$

$$\tau(t) = X(t) + X(t-\tau(t)) \tag{1.2}$$

which describes the straight-line motion (along the X-axis) of two charged particles of equal mass m having the charges e_1, e_2 (here $k = e_1 e_2/m$), which is symmetric with respect to the origin. Thus, the coordinates of the particles are $X(t)$ and $-X(t)$, respectively. The radiation damping force is not considered (see, however, Section 6). The system (1.1), (1.2) can be obtained by means of the usual expression for the Lorentz force using the Maxwell equations.

The two-body problem of classical electrodynamics has been studied elsewhere (Driver, 1963; Driver & Norris, 1967; Driver, 1969; Driver, 1970). These papers deal with the qualitative behavior of two charged particles in the case of a straight-line motion. The numerical investigation of this problem has been carried out by Kasher & Schwebel (1971) and by Hushilt et al. (1973). In this paper we study the question of uniqueness of solutions of the system (1.1), (1.2) (equal mass case) and present some estimates of the solutions.

It should be noted that in the general case the solutions of the equations with deviating arguments [these include equation (1.1)] are not specified uniquely by the instantaneous data given at the initial moment $t = t_0$ (Elsgolts, 1966). However, if one demands that $X(t)$ must satisfy the equations of motion also for $t \in (-\infty, t_0]$ (not only for $t \geqslant t_0$), one may expect that this prescription provides the uniqueness of solutions. This formulation of the problem in the case of the system (1.1), (1.2) was considered by Driver (1970) (for $k > 0$; see also Driver, 1969). His result may be reformulated as follows: If a certain quantity, which can be roughly estimated as $k/X_0V_0^2$ (X_0, V_0 are the initial data at $t = t_0$; $X_0 > 0$, $V_0 < 0$), is sufficiently small, then the solution of (1.1), (1.2) exists and it is unique. This result cannot be applied in the domain of initial data where $V_0 \geqslant 0$ or $k/X_0V_0^2$ is comparable with unity. We shall extend the validity of the uniqueness theorem to this domain. The uniqueness (and the existence) of solutions will be proved under the condition that $k/X(t)$ is sufficiently small for all $t \leqslant t_0$. Note that if $k/X(t) \gtrsim 1$, there is some doubt concerning the validity of the equations used. In the case $V_0 \geqslant 0$ the condition stated above is guaranteed when the classical energy at the initial moment $V_0^2 + k/X_0$ is sufficiently small. The case of attractive interactions ($k < 0$) is also considered; however, our treatment does not embrace all possible nonrelativistic motions. The estimates will also be obtained (Section 3) which allow one to give an approximate description of motion.

2. *The Case $k > 0$: Existence of Solutions of the Equation (1.2)*

We consider the solutions satisfying the inequalities

$$X(t) > 0 \tag{2.1}$$

$$|\dot{X}(t)| < 1 \tag{2.2}$$

In order to study the existence of solutions of (1.2), considered along with equation (1.1), with respect to $\tau(t)$, one must define the system (1.1), (1.2) when the solution of (1.2) does not exist. For physical reasons we write

$$\frac{\ddot{X}(t)}{[1 - \dot{X}^2(t)]^{3/2}} = \int d\tau\, \delta[\tau - X(t) - X(t-\tau)]\, \frac{k}{\tau}\, 1 - \dot{X}(t-\tau) \tag{2.3}$$

(δ is the Dirac function) because just this equation is obtained directly from the Maxwell equations, and the system (1.1), (1.2) follows from (2.3) as a special case when the solution of (1.2) exists. When this solution is absent the right-hand side of (2.3) vanishes, the interaction does not reach the particles, and they move with uniform velocity. We shall first confine ourselves to continuous solutions assuming no differentiability, because the singularity of the right-hand side of (2.3) may take place at the moment when the solution of (1.2) disappears. In this case the condition (2.2) transforms to

$$\operatorname{Var}(X, t, \Delta t) < 1, \qquad \Delta t \neq 0 \tag{2.4}$$

where

$$\operatorname{Var}(X, t, \Delta t) = \frac{X(t + \Delta t) - X(t)}{\Delta t}$$

Suppose (for contradiction) that the solution of (1.2) does not exist for $t = t_0$. Then for $\forall t' < t_0$ the solution does not exist either. Indeed, if $\exists \tau'$ such that

$$\tau' = X(t') + X(t' - \tau')$$

then taking into account (2.4) we obtain for $t' < t_0$

$$f(\alpha) > 0 \qquad \text{for } \alpha = t' - \tau'$$
$$f(\alpha) < 0 \qquad \text{for } \alpha = t_0$$

where

$$f(\alpha) = t_0 - \alpha - X(\alpha) - X(t_0)$$

Then $\exists \alpha_0 : f(\alpha_0) = 0$ and $\tau_0 = t_0 - \alpha_0$ is the solution of (1.2) for $t = t_0$, i.e., we have the contradiction. Thus for $t \in (-\infty, t_0]$ the solution of (1.2) does not exist and the right-hand side of (2.3) vanishes. From (2.3), (2.4) we have

$$\operatorname{Var}(X, t, \Delta t) = \text{const} < 1, \qquad \Delta t \neq 0, \quad t \leqslant t_0 \tag{2.5}$$

Using (2.5) it is easy to prove the existence of the solution for $t \leqslant t_0$, i.e., we again have the contradiction.

We can now formulate the following lemma: if $X(t) \in C_{(-\infty, t_0]}$ satisfying (2.4) is the solution of (2.3), then the solution of (1.2) with respect to $\tau(t)$ exists for all $t \leqslant t_0$.

We do not define here in what sense the continuous function $X(t)$ may be the solution of (2.3), because this is not essential for further considerations and one may choose $X(t)$ to be a differentiable function. According to Driver (1970), if the solution of (1.2) exists, then $\exists\, c$:

$$|\dot{X}(t)| \leqslant c < 1 \tag{2.6}$$

In analogy to Driver (1963), we obtain, using (2.6), the uniqueness of $\tau(t)$ and

$$\frac{2X(t)}{1 + c} \leqslant \tau(t) \leqslant \frac{2X(t)}{1 - c} \tag{2.7}$$

$$|\tau(X, t) - \tau(X', t)| \leqslant \frac{|\delta X(t)| + |\delta X(t - \tau(X, t))|}{1 - c} \tag{2.8}$$

where

$$\delta X(t) = X(t) - X'(t)$$

$$\frac{d\tau(t)}{dt} = \frac{\dot{X}(t) + \dot{X}(t - \tau(t))}{1 + \dot{X}(t - \tau(t))} \tag{2.9}$$

3. *The Estimates of Solutions ($k > 0$)*

Let $X(t)$ be a solution of (2.3). One can easily see that $\dot{X}(t) < 0$ for $t \to -\infty$, $\dot{X}(t) > 0$ for $t \to \infty$, and there exists a single point t where $X(t) = 0$ (the turning point).

a. Consider the motion on that part of the trajectory where $\dot{X}(t) \leqslant 0$. Multiplying (1.1) by $\dot{X}(t)$ and taking into account (2.9) we obtain

$$\frac{d}{dt} \frac{1}{\sqrt{[1 - \dot{X}^2(t)]}} + \frac{k}{\tau(t)} \geqslant 0$$

Therefore

$$\frac{1}{\sqrt{(1 - \alpha^2)}} \leqslant \frac{1}{\sqrt{[1 - \dot{X}^2(t)]}} + \frac{k}{2X(t)} \tag{3.1}$$

where

$$\alpha = \lim_{t \to -\infty} \dot{X}(t)$$

This inequality gives the upper estimate for α by means of $X(t)$, $\dot{X}(t)$. Evidently, $\dot{X}(t) > -\alpha$. Multiplying (1.1) by $\dot{X}(t)$ and using (2.7) we have

$$\frac{1}{\sqrt{[1 - \dot{X}^2(t)]}} + \frac{k(1 - \alpha)^2}{4X(t)} \leqslant \frac{1}{\sqrt{(1 - \alpha^2)}} \tag{3.2}$$

$$\frac{1}{\sqrt{[1 - \dot{X}^2(t)]}} + \frac{k(1 + \alpha)^3}{4(1 - \alpha)X(t)} \geqslant \frac{1}{\sqrt{(1 - \alpha^2)}} \tag{3.3}$$

For $\alpha \ll 1$ the relations (3.2), (3.3) give an approximate picture of the phase trajectory distribution.

After the simple transformations we obtain from (3.2), (3.3)

$$\dot{X}^2(t) + \kappa/X(t) \leqslant E \tag{3.4}$$

$$\dot{X}^2(t) + \kappa'/X(t) \leqslant E' \tag{3.5}$$

where

$$\kappa = (k/2)(1 - \alpha)^2, \qquad E = 2[(1 - \alpha^2)^{-1/2} - 1]$$

$$\kappa' = (k/2)(1 + \alpha)^4 (1 - \alpha^2)^{1/2}, \qquad E' = 2(1 - \alpha^2)^{3/2}[(1 - \alpha^2)^{-1/2} - 1]$$

Then from (3.4) it follows

$$\int_{X(t_0)}^{X(t)} \frac{dX}{\sqrt{(E - \kappa/X)}} \leqslant t_0 - t, \qquad t_0 \geqslant t$$

On performing the integration we have

$$E^{-1}\left\{X(t)\sqrt{\left[E - \frac{\kappa}{X(t)}\right]} - X(t_0)\sqrt{\left[E - \frac{\kappa}{X(t_0)}\right]}\right.$$
$$\left. + \frac{\kappa}{\sqrt{E}}\ln\left[\frac{X(t)\{\sqrt{[E - \kappa/X(t)]} + \sqrt{E}\}}{X(t)\{\sqrt{[E - \kappa/X(t_0)]} + \sqrt{E}\}}\right]\right\} \leqslant t_0 - t, \qquad t_0 \geqslant t \tag{3.6}$$

By analogy with (3.6)

$$\frac{1}{E'}\left\{X(t)\sqrt{\left[E' - \frac{\kappa'}{X(t)}\right]} - X(t_0)\sqrt{\left[E' - \frac{\kappa'}{X(t_0)}\right]}\right.$$
$$\left. + \frac{\kappa'}{\sqrt{E'}}\ln\frac{X(t)\{\sqrt{[E' - \kappa'/X(t)]} + \sqrt{E'}\}}{X(t_0)\{\sqrt{[E' - \kappa'/X(t_0)]} + \sqrt{E'}\}}\right\} \geqslant t_0 - t, \qquad t \leqslant t_0 \tag{3.7}$$

For $\alpha \ll 1$ the inequalities (3.6), (3.7) may be used to define implicitly an approximate solution of (1.1), (1.2) and give the error of the approximation. We shall derive another relation which may be useful when α is comparable with unity, but $k/X(t) \ll 1$ for $t \leqslant t_0$. Making use of the identies

$$X(t - \tau) = X(t) - \tau\dot{X}(t) + \tau^2 \int_0^1 ds \int_0^s ds' \ddot{X}(t - \tau s') \tag{3.8}$$

$$\dot{X}(t - \tau) = \dot{X}(t) - \tau \int_0^1 ds \ddot{X}(t - \tau s) \tag{3.9}$$

and transforming (1.1) we obtain

$$\frac{\ddot{X}(t)}{[1 - \dot{X}^2(t)]^{3/2}} = \frac{k[1 - \dot{X}^2(t)]}{4X^2(t)} + J(X, t)$$

where

$$J(X, t) = \frac{k}{\tau^2(t)}\frac{1 - \dot{X}(t - \tau(t))}{1 + \dot{X}(t - \tau(t))} - \frac{k}{\tau_a^2(t)}\frac{1 - \dot{X}(t)}{1 + \dot{X}(t)}$$

$$\tau_a(t) = 2X(t)/[1 + \dot{X}(t)]$$

Multiplying by $X(t)$ we have

$$\frac{d}{dt}\left\{\frac{1}{3[1 - \dot{X}^2(t)]^{3/2}} + \frac{k}{4X(t)}\right\} = K(X, t) \tag{3.10}$$

where

$$K(X,t)=\frac{\dot X(t)}{1-\dot X^2(t)}\left\{\frac{k}{\tau(t)}\,\frac{2\int\limits_0^1 ds\,\ddot X(t-\tau(t)s)}{[1+\dot X(t-\tau(t))]\,[1+\dot X(t)]}\right.$$

$$\left.+\frac{1-\dot X(t)}{1+\dot X(t)}\left[\frac{k}{\tau(t)}+\frac{k\tau_a(t)}{\tau^2(t)}\right]\frac{\int\limits_0^1 ds\int\limits_0^s ds'\ddot X(t-s'\tau_a(t))}{1-\int\limits_0^1 ds\dot X(t-\tau_a(t)-s[\tau(t)-\tau_a(t)])}\right\} \tag{3.11}$$

We use here the relation

$$\tau(t)-\tau_a(t)=\frac{\tau_a^2(t)\int\limits_0^1 ds\int\limits_0^s ds'\ddot X(t-s'\tau_a(t))}{1+\int\limits_0^1 ds\dot X(t-\tau_a(t)-s[\tau(t)-\tau_a(t)])} \tag{3.12}$$

which is obtained by making use of a transformation analogous to (3.8), (3.9) in the equation (1.2).

From (1.1) for $\dot X(t)\leqslant 0$ it follows that

$$\ddot X(t-s\tau_a\ (t))\leqslant k(1+\alpha)^3/4(1-\alpha)X^2(t),\qquad s\in[0,1] \tag{3.13}$$

$$\ddot X(t-s\tau_a(t))\geqslant k(1-\alpha)^4(1-\alpha^2)^{1/2}/4X^2(t) \tag{3.14}$$

These esimates persist if we substitute $\tau_a(t)$ for $\tau(t)$. Using (3.13), (3.14) one one can estimate (3.11):

$$kC_1\dot X(t)/X^3(t)\leqslant K(X,t)\leqslant kC_2\dot X(t)/X^3(t) \tag{3.15}$$

where C_1, C_2 are the constants depending upon α and bounded for $\alpha\to 0$. Taking advantage of (3.10) we have

$$\left|\frac{1}{3[1-\dot X^2(t)]^{3/2}}+\frac{k}{4X(t)}-\frac{1}{3[1-\dot X^2(t_0)]^{3/2}}-\frac{k}{4X(t_0)}\right|\leqslant\frac{k}{X^2(t)}\frac{(|C_1|+|C_2|)}{2} \tag{3.16}$$

This relation gives the estimate of the error of the approximate expression describing the distribution of the phase curves of the equation (2.3), which is obtained by putting the left-hand side of (3.16) equal to zero. The relative error of the approximation is defined by the quantity sup $k/X(t)$. Certainly, all the relations remain valid when $k/X(t)$ is comparable with unity. However, the approximation defined by (3.16) is good only if sup $k/X(t)\ll 1$. It is not necessary here that $\alpha \leqslant 1$, but if all the trajectory is considered [all $t \in (-\infty, \infty)$], then $\alpha \leqslant 1$ owing to (3.1).

b. Now consider $X(t)$ on the right-hand side of the turning point [$X(t) \geqslant 1$]. The calculations are quite similar to (a); one should only remember that $\dot{X}(t - \tau(t))$ changes its sign, though $X(t) \geqslant 0$.

Let $\dot{X}(t_1) = 0$. Using (3.1) for $t \leqslant t_1$ we have

$$1/\sqrt{(1-\alpha^2)} \leqslant 1 + k/2X(t_1) \tag{3.17}$$

Taking into account the sign of $\dot{X}(t - \tau(t))$ we have for $t \geqslant t_1$

$$\frac{1}{\sqrt{[1-\dot{X}^2(t)]}} + \frac{k}{X(t)}\,\frac{1+\alpha}{1-\alpha} < 1 + \frac{k(1+\alpha)}{X(t_1)(1-\alpha)} \tag{3.18}$$

Comparing (3.17) to (3.18) one can see that the constant c of (2.6) can be defined as follows

$$\frac{1}{\sqrt{(1-c^2)}} = 1 + \frac{k}{X(t_1)}\,\frac{1+\alpha}{1-\alpha} \tag{3.19}$$

The following considerations are analogous to (a). In particular, the relations corresponding to (3.6), (3.7) differ from them only by the sign of inequalities and by the value of the constants E, E', κ, κ'.

Later we shall use the estimate

$$\frac{\dot{X}(t)}{\sqrt{[1-\dot{X}^2(t)]}} + \frac{\alpha}{(1-\alpha^2)^{1/2}} \geqslant \int_0^1 ds\, \frac{k(1-c)}{\tau^2(s)(1+c)} \tag{3.20}$$

which is valid for all $t \in (-\infty, \infty)$ and is obtained by integration of (1.1).

4. *Existence and Uniqueness of the Problem with Point Initial Data ($k > 0$)*

We shall consider first the uniqueness. Let $X(t)$, $X'(t)$ be the solutions of (2.3) satisfying

$$X(t_0) = X(t_0) = X_0, \qquad \dot{X}(t_0) = \dot{X}'(t_0) = \dot{X}_0 \tag{4.1}$$

Following Driver (1969), we start from the "backwards" solutions ($t \leqslant t_0$).

a. Consider first the case $\dot{X}_0 \leqslant 0$. Using (3.1) one can give an upper estimate for α and c in terms of $X_0, \dot{X}_0$. Taking advantage of the relation

$$|\tau(t) - \tau'(t)| \leqslant \frac{1}{1-\alpha}\left\{2\int_{t_0}^{t} ds\,|\delta\dot{X}(s)| + \tau(t)\int_0^1 ds\,|\delta\dot{X}(t - s\tau(t))|\right.$$

where $\delta\dot{X}(t) = \dot{X}(t) - \dot{X}'(t)$, we obtain, after some calculations

$$\|\delta\dot{X}\|_t^{t_0} \leqslant \int_t^{t_0} ds\,\frac{kA_1\|\delta\dot{X}\|_{-\infty}^{t_0}}{\rho^2(s)} + \int_t^{t_0} ds\,\frac{kA_2}{\rho^3(s)}\int_s^{t_0} ds'\,\|\delta\dot{X}\|_{s'}^{t_0}$$

where

$$\rho(s) = \min \{\tau(s), \tau'(s)\}, \qquad \|f\|_b^a = \sup |f(t)|, \qquad t \in [a, b]$$

the constants A_1, A_2 depending on α.

Using relations analogous to (3.7) we have

$$\frac{t_0 - t}{\min (X(t), X'(t))} \leqslant \frac{1}{\sqrt{E'}} \left\{1 + \frac{\kappa'}{X_0 E'} \left[\frac{1}{2e} + \ln 2\right]\right\} \stackrel{df}{=} A_3 \tag{4.2}$$

Here $e = 2{,}718 \cdots (\ln e = 1)$

Using (3.20), (4.2) we find, solving the inequality written above with respect to $\| \delta \dot{X} \|_t^{t_0}$, that

$$\| \delta \dot{X} \|_t^{t_0} \leqslant L \| \delta \dot{X} \|_{-\infty}^{t_0}$$

where

$$L = \frac{\alpha A_1 (1 + \alpha)}{(1 - \alpha)\sqrt{(1 - \alpha^2)}} \exp \frac{\alpha (1 + \alpha)^2 A_2 A_3}{2(1 - \alpha)\sqrt{(1 - \alpha^2)}}$$

We see that if

$$L < 1 \tag{4.3}$$

then the solution is unique: $\delta \dot{X}(t) \equiv 0$.

We shall estimate the constants A_1, A_2, A_3 in the nonrelativistic case when $\dot{X}_0^2 + k/2X_0 \ll 1$. The results are

$$\alpha^2 \approx E \approx E' \approx \dot{X}_0^2 + k/2X_0$$

$$A_1 \approx 4, \qquad A_2 \approx 4, \qquad A_3 \approx \alpha^{-1}(1 + 1/2e + \ln 2)$$

Then $L \approx 16 e^{2+1/e}$ and (4.3) is satisfied for $\dot{X}_0^2 + k/2X_0$ sufficiently small.

If $|\dot{X}_0|$ is comparable with unity, but $k/X_0 \ll 1$, the condition (4.3) is ineffective. In order to find a more convenient condition we write using (4.2)

$$\int_{-\infty}^{t_0} ds \, k\tau^{-2}(s) \leqslant \frac{k(1 + \alpha)^2 A_3}{4X_0}$$

Making use of this estimate instead of (3.20) one obtains

$$L = O\left(\frac{k}{X_0 |\dot{X}_0|}\right) \exp O\left(\frac{k}{X_0 |\dot{X}_0^2|}\right) \sim O\left(\frac{k}{X_0}\right)$$

Here $O(z)$ is the quantity of the order of z.

b. Consider now $X_0 \geqslant 0$. It will be essential for our reasoning that $\dot{X}_0^2 + k/2X_0$ be sufficiently small. For simplicity we shall confine ourselves to the requirement $\dot{X}_0^2 + k/2X_0 \ll 1$, which is always satisfied in the nonrelativistic case. In this case $|\dot{X}(t)| \leqslant c \ll 1$, and $\dot{X}(t)$, $\dot{X}'(t)$ may differ only by the quantity of the order $O(c) \max(|\dot{X}(t)|, |\dot{X}'(t)|)$. To prove this one should use

(3.6), (3.7) and their analogs for $\dot{X}(t) \geqslant 0$. In this connection we shall write down the constants in the subsequent estimates approximately, neglecting the higher-order terms in c.

Let $\dot{X}(t_1) = 0$, $t_1 \leqslant t_0$. Denote $X_1 = X(t_1)$. On account of the invariance of (2.3) with respect to time translations, one can choose the parameter t for $X'(t)$ in such a way that $\dot{X}'(t_1) = 0$, the phase trajectories being preserved. In this case $X'(t)$ will satisfy the initial conditions (4.1) not at $t = t_0$, but at some other instant $t = t_0'$. Denote $X'(t_1) = X_1' = X_1 + \delta X_1$. If $\delta X_1 = 0$, the first part of this section yields $X(t) \equiv X'(t)$ for $t \leqslant t_1$, and $X(t) \equiv X'(t)$ for $t \geqslant t_1$ follows in virtue of the known theorem (Driver, 1963). We shall suppose $\delta X_1 \neq 0$ and show that this leads to contradiction.

By analogy to the above calculations, estimating $\delta \dot{X}(t)$ first for $t \leqslant t_1$ and then for $t \geqslant t_1$, we obtain

$$\| \delta \dot{X} \|_{-\infty}^{\infty} \leqslant 8e^{2+1/e}\alpha | \delta X_1 | / X_1 \tag{4.4}$$

Using (3.10) one can write

$$\frac{1}{3[1 - \dot{X}^2(t)]^{3/2}} + \frac{k}{4X(t)} = \frac{1}{3} + \frac{k}{4X_1} + \int_{t_1}^{t} ds\, K(X, s) \tag{4.5}$$

When $X(t)$ varies to $X'(t)$ we have

$$\int_{t_1}^{t_0} ds\, K(X, s) - \int_{t_1}^{t_0'} ds\, K(X', s) = \int_{\dot{X}(t_0')}^{\dot{X}(t_0)} K(X, s)\, \frac{d\dot{X}(s)}{\ddot{X}(s)} + \int_{t_1}^{t_0'} ds\, \delta K(X, s) \tag{4.6}$$

Estimating C_2 in (3.15) we obtain, after rather lengthy calculations,

$$\int_{t_0}^{t} ds\, | \delta K(X, s) | \leqslant \frac{\alpha^2 k}{X_1}\, 84 e^{2+1/e}\, \frac{|\delta X_1|}{X_1} \tag{4.7}$$

$$\int_{X(t_0')}^{\dot{X}(t_0)} d\dot{X}(s)\, \frac{K(X, s)}{\ddot{X}(s)} \leqslant 12 e^{2+1/e}\, \frac{\alpha^2 k |\delta X_1|}{X_1^2} \tag{4.8}$$

It should be remembered that we write down the constants approximately. Taking into account that $\dot{X}(t_0) = \dot{X}'(t_0')$ and (4.7), (4.8), we have from (4.5)

$$\frac{k}{X(t_0)} - \frac{k}{X'(t_0')} = \left(\frac{k}{X_1} - \frac{k}{X_1'}\right)[1 - O(\alpha^2)] \tag{4.9}$$

When $\alpha^2 \approx \dot{X}_0^2 + k/2X_0$ is sufficiently small, the relation (4.9) contradicts (4.1) for $\delta X_1 \neq 0$. This proves the uniqueness for $\dot{X}_0 \geqslant 0$. Here α may be considered sufficiently small if $\alpha \lesssim 10^{-3}$ [in this case $L < 1$, $O(\alpha^2) < 1$]. Thus, the uniqueness of the "backwards" solutions is proved for arbitrary sign of $\dot{X}_0$. The uniqueness for the global solutions follows from this according to the result of Driver (1963).

c. The existence of solutions satisfying (4.1). Because the proof is very simple we give only its outline. Following Driver (1969) one can prove the existence of solutions with initial data $X_1, \dot{X}_1$, for which $k/X_1\dot{X}_0^2 \ll 1, \dot{X}_1 < 0$. Using (3.1)-(3.3) and their analogs at the right-hand side of the turning point, it is easy to prove that there exist initial data (for $t = t_1$)$X_1', \dot{X}_1', \dot{X}_1''$, such that the point $(X_0, \dot{X}_0)$ of the phase plane lies between the phase trajectories $(X(t), \dot{X}(t))$ and $(X''(t), \dot{X}''(t))$. In the domain of the phase plane (X, Y), where $k/XY^2 \ll 1$, $Y < 0$, the continuous dependence of the "backwards" solutions upon $X_1, \dot{X}_1$ can be easily proved and so we infer the continuous dependence of the global solutions upon $X_1, \dot{X}_1$, making use of the result of Driver (1963). From this it follows that there exists the trajectory $(X_0(t), \dot{X}_0(t))$ which passes through the point $X_0, \dot{X}_0$ at some $t = t_2$. This proves the existence of solutions satisfying (4.1), because the invariance of (2.3) with respect to time translations allows us to change $t_2 \to t_0$.

We shall sum up the results obtained.

Theorem. The equation (2.3) admits the unique global solution $X(t)$ $[t \in (-\infty, \infty)]$ corresponding to the data (4.1) in the set of functions of $C^2_{(-\infty,\infty)}$ satisfying the conditions (2.1), (2.2), if $(\dot{X}_0^2 + k/2X_0)$ is sufficiently small.

5. *Opposite Sign of the Charges* $(k < 0)$

If the charges of the particles have an opposite sign, the situation differs from that considered in Section 4. It is easy to see that if $\dot{X}(t_0) \geqslant 0$, then $X(t)$ cannot be extended to all $t \leqslant t_0$, because in this case

$$\exists t_1 : \lim_{t \to t_1 + 0} X(t) = 0$$

Taking into account that $\dot{X}(t) < 0$ we have, in analogy with (3.2), (3.3),

$$\frac{1}{\sqrt{[1 - \dot{X}^2(t)]}} - \frac{|k|[1 - |\dot{X}(t_0)|]}{4X(t)} \geqslant \frac{1}{\sqrt{(1-\alpha^2)}}, \qquad \alpha = -\lim_{t \to -\infty} \dot{X}(t) \tag{5.1}$$

This imposes a certain restriction on the possible initial data and the solutions for $t \leqslant t_0$. The inequality

$$\frac{1}{\sqrt{[1 - \dot{X}^2(t)]}} - \frac{|k|}{4X(t)} \leqslant \frac{1}{\sqrt{(1-\alpha^2)}} \tag{5.2}$$

obtained in an analogous manner gives the lower estimate for α. Further considerations are analogous to Sections 3, 4a, and 4c. The condition for the uniqueness is

$$C_1 \sqrt{\frac{|k|}{X(t_0)}} \exp C_2 \left[\frac{|\dot{X}(t_0)|^2 X(t_0)}{|k|} - \frac{1}{2}\right]^{-1/2} < 1$$

where the constants C_1, C_2 are bounded for small α. Here, besides the condition that $|k/X_0|$ should be sufficiently small one must demand $\alpha \gtrsim O(\sqrt{|k/X_0|})$.

6. *Concluding Remarks*

From the results established so far we infer that in a certain domain of the phase plane the phase trajectories do not cross each other. On account of this one may state that there exists the Newtonian equation (of the second order without deviating arguments) which is equivalent in a certain domain to the system (1.1), (1.2) [or (2.3)].

In the above treatment we neglected the radiation damping force. Had we taken the usual expression for this force (containing the third derivative) in the equations of motion, this would have changed significantly our considerations and led to certain mathematical troubles. These troubles arise from the assumption that the particles have a pointlike structure. However, in our opinion it is more reasonable to admit an extended structure of the particles. In this case one can obtain another expression for the radiation damping force which is analogous to that of Zhdanov (1974). Detailed investigation shows that this expression has properties analogous to that of $J(X, t)$ (see (3.10), (3.11) multiplied by $X(t)/R$, where R is the radius of the particles. To write down an explicit expression one must know the details of the internal structure of the particles and the forces which provide stability for the particles. Our assertion concerns the case when these forces have a structure similar to that of an electromagnetic interaction. If $k/R \ll 1$, we can preserve the basic lines of our proof and the results of Sections 4 and 5 remain valid.

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